

# A perturbative approach for the study of compatibility between nonminimally coupled gravity and Solar System experiments

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**Abstract.** We develop a framework for constraining a certain class of theories of nonminimally coupled (NMC) gravity with Solar System observations.

## 1. Introduction

We consider the possibility of constraining a class of theories of nonminimally coupled gravity [1] by means of Solar System experiments. NMC gravity is an extension of  $f(R)$  gravity where the action integral of General Relativity (GR) is modified in such a way to contain two functions  $f^1(R)$  and  $f^2(R)$  of the space-time curvature  $R$ . The function  $f^1(R)$  has a role analogous to  $f(R)$  gravity, and the function  $f^2(R)$  yields a nonminimal coupling between curvature and the matter Lagrangian density. For other NMC gravity theories and their potential applications, see, e.g., [2, 3, 4, 5, 6].

NMC gravity has been applied to several astrophysical and cosmological problems such as dark matter [7, 8], cosmological perturbations [9], post-inflationary reheating [10] or the current accelerated expansion of the Universe [11].

In the present communication, by extending the perturbative study of  $f(R)$  gravity in [12], we discuss how a general framework for the study of Solar System constraints to NMC gravity can be developed. The approach is based on a suitable linearization of the field equations of NMC gravity around a cosmological background space-time, where the Sun is considered as a perturbation. Solar System observables are computed, then we apply the perturbative approach to the NMC model by Bertolami, Frazão and Páramos [11], which constitutes a natural extension of  $1/R^n$  ( $n > 0$ ) gravity [13] to the non-minimally coupled case. Such a NMC gravity model is able to predict the observed accelerated expansion of the Universe. We show that, differently from the pure  $1/R^n$  gravity case, the NMC model cannot be constrained by this perturbative method so that it remains, in this respect, a viable theory of gravity. Further details about the subject of the present communication can be found in the manuscript [14].

## 2. NMC gravity model

We consider a gravity model with an action functional of the type [1],

$$S = \int \left[ \frac{1}{2} f^1(R) + [1 + f^2(R)] \mathcal{L}_m \right] \sqrt{-g} d^4x,$$

where  $f^i(R)$  ( $i = 1, 2$ ) are functions of the Ricci scalar curvature  $R$ ,  $\mathcal{L}_m$  is the Lagrangian density of matter and  $g$  is the metric determinant. By varying the action with respect to the metric we get the field equations

$$(f_R^1 + 2f_R^2 \mathcal{L}_m) R_{\mu\nu} - \frac{1}{2} f^1 g_{\mu\nu} = (1 + f^2) T_{\mu\nu} + \nabla_{\mu\nu} (f_R^1 + 2f_R^2 \mathcal{L}_m), \quad (1)$$

where  $f_R^i \equiv df^i/dR$  and  $\nabla_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \square$ . We describe matter as a perfect fluid with negligible pressure: the Lagrangian density of matter is  $\mathcal{L}_m = -\rho$  and the trace of the energy-momentum tensor is  $T = -\rho$ . We write  $\rho = \rho^{\text{cos}} + \rho^s$ , where  $\rho^{\text{cos}}$  is the cosmological mass density and  $\rho^s$  is the Sun mass density.

We assume that the metric which describes the spacetime around the Sun is a perturbation of a flat Friedmann-Robertson-Walker (FRW) metric with scale factor  $a(t)$ :

$$ds^2 = -[1 + 2\Psi(r, t)] dt^2 + a^2(t) ([1 + 2\Phi(r, t)] dr^2 + r^2 d\Omega^2),$$

where  $|\Psi(r, t)| \ll 1$  and  $|\Phi(r, t)| \ll 1$ . The Ricci curvature of the perturbed spacetime is expressed as the sum

$$R(r, t) = R_0(t) + R_1(r, t),$$

where  $R_0$  denotes the scalar curvature of the background FRW spacetime and  $R_1$  is the perturbation due to the Sun. Following Ref. [12], we linearize the field equations assuming that

$$|R_1(r, t)| \ll R_0(t), \quad (2)$$

both around and inside the Sun. This assumption means that the curvature  $R$  of the perturbed spacetime remains close to the cosmological value  $R_0$  inside the Sun. In GR such a property of the curvature is not satisfied inside the Sun. However, for  $f(R)$  theories which are characterized by a small value of a suitable mass parameter (see next section), condition (2) can be satisfied. For instance, the  $1/R^n$  ( $n > 0$ ) gravity model [13] satisfies condition (2), as shown in [12, 15].

Eventually, we assume that functions  $f^1(R)$  and  $f^2(R)$  admit a Taylor expansion around  $R = R_0$  and that terms nonlinear in  $R_1$  can be neglected in the expansion. We use the notation introduced by [12] (for  $i = 1, 2$ ):

$$f_0^i \equiv f^i(R_0) \quad , \quad f_{R0}^i \equiv \frac{df^i}{dR}(R_0) \quad , \quad f_{RR0}^i \equiv \frac{d^2 f^i}{dR^2}(R_0).$$

## 3. Solution of the linearized field equations

The details of the following computations can be found in the paper [14]. First we linearize the trace of the field equations (1). Using condition (2), we neglect  $O(R_1^2)$  contributions but we keep the cross-term  $R_0 R_1$ . Introducing the potential  $U = (f_{RR0}^1 + 2f_{RR0}^2 \mathcal{L}_m) R_1$ , we get

$$\nabla^2 U - m^2 U = -\frac{1}{3} (1 + f_0^2) \rho^s + \frac{2}{3} f_{R0}^2 \rho^s R_0 + 2\rho^s \square f_{R0}^2 + 2f_{R0}^2 \nabla^2 \rho^s,$$

where  $m^2$  denotes the mass parameter

$$m^2(r, t) = \frac{1}{3} \left[ \frac{f_{R0}^1 - f_{R0}^2 \mathcal{L}_m}{f_{RR0}^1 + 2f_{RR0}^2 \mathcal{L}_m} - R_0 - \frac{3\square (f_{RR0}^1 - 2f_{RR0}^2 \rho^{\text{cos}}) - 6\rho^s \square f_{RR0}^2}{f_{RR0}^1 + 2f_{RR0}^2 \mathcal{L}_m} \right]. \quad (3)$$

When  $f^2(R) = 0$  we recover the mass formula of  $f(R)$  gravity theory found in [12]. In the following we assume that  $|mr| \ll 1$  at Solar System scale. Under this assumption the solution for  $R_1$  outside the Sun is given by

$$R_1(r, t) = \left[ \frac{-\frac{1}{3}(1 + f_0^2) + \frac{2}{3}f_{R0}^2 R_0 + 2\Box f_{R0}^2}{4\pi(2f_{R0}^2 \rho^{\cos} - f_{R0}^1)} \right] \frac{M_S}{r}, \quad (4)$$

where  $M_S$  is the mass of the Sun. Then we linearize the field equations (1) obtaining

$$\begin{aligned} (f_{R0}^1 + 2f_{R0}^2 \mathcal{L}_m) \left( \nabla^2 \Psi + \frac{1}{2} R_1 \right) - \nabla^2 [(f_{R0}^1 + 2f_{R0}^2 \mathcal{L}_m) R_1] &= (1 + f_0^2) \rho^s - 2f_{R0}^2 \nabla^2 \rho^s, \\ (f_{R0}^1 + 2f_{R0}^2 \mathcal{L}_m) \left( -\frac{d^2 \Psi}{dr^2} + \frac{2}{r} \frac{d\Psi}{dr} \right) - \frac{1}{2} f_{R0}^1 R_1 + \frac{2}{r} f_{R0}^1 \frac{dR_1}{dr} + \frac{4}{r} f_{R0}^2 \frac{\partial(\mathcal{L}_m R_1)}{\partial r} &= \frac{4}{r} f_{R0}^2 \frac{d\rho^s}{dr}. \end{aligned}$$

Using the divergence theorem and the solution (4) for  $R_1$ , from the first equation we obtain the function  $\Psi$  outside of the Sun:

$$\Psi(r, t) = -\frac{2}{3r} (1 + f_0^2 + f_{R0}^2 R_0 + 3\Box f_{R0}^2) \int_0^{R_S} \frac{\rho^s(x)}{f_{R0}^1 + 2f_{R0}^2 \mathcal{L}_m(x)} r^2 dr,$$

where  $R_S$  is the radius of the Sun. If the following condition is satisfied,

$$|2f_{R0}^2| \rho^s(r) \ll |f_{R0}^1 - 2f_{R0}^2 \rho^{\cos}(t)|, \quad r \leq R_S, \quad (5)$$

then the function  $\Psi$  is a Newtonian potential:

$$\Psi(r, t) = -\frac{GM_S}{r}, \quad G(t) = \frac{1 + f_0^2 + f_{R0}^2 R_0 + 3\Box f_{R0}^2}{6\pi(f_{R0}^1 - 2f_{R0}^2 \rho^{\cos})} \quad r \geq R_S, \quad (6)$$

where  $G(t)$  is an effective gravitational constant. Since  $G$  depends on slowly varying cosmological quantities we have  $G(t) \simeq \text{constant}$ , so that  $\Psi(r, t) \simeq \Psi(r)$ .

The solution for the function  $\Phi$  is computed from the second of the linearized field equations, and we obtain  $\Phi(r) = -\gamma \Psi(r)$ , where the PPN parameter  $\gamma$  depends on cosmological quantities and it is given by

$$\gamma = \frac{1}{2} \left[ \frac{1 + f_0^2 + 4f_{R0}^2 R_0 + 12\Box f_{R0}^2}{1 + f_0^2 + f_{R0}^2 R_0 + 3\Box f_{R0}^2} \right].$$

When  $f^2(R) = 0$  we find the known result  $\gamma = 1/2$  which holds for  $f(R)$  gravity theories which satisfy the condition  $|mr| \ll 1$  and condition  $|R_1| \ll R_0$ , as it has been shown in [12]. The  $1/R^n$  ( $n > 0$ ) gravity theory [13], where  $f(R)$  is proportional to  $(R + \text{constant}/R^n)$ , is one of such theories that, consequently, have to be ruled out by Cassini measurement.

#### 4. Application to a NMC cosmological model

We consider the NMC gravity model proposed in [11] to account for the observed accelerated expansion of the Universe:

$$f^1(R) = 2\kappa R, \quad f^2(R) = \left( \frac{R}{R_n} \right)^{-n}, \quad n > 0, \quad (7)$$

where  $\kappa = c^4/16\pi G_N$ ,  $G_N$  is Newton's gravitational constant, and  $R_n$  is a constant. This model yields a cosmological solution with a negative deceleration parameter  $q < 0$ , and the scale factor

$a(t)$  of the background metric follows the temporal evolution  $a(t) = a_0 (t/t_0)^{2(1+n)/3}$ , where  $t_0$  is the current age of the Universe. Using the properties of the cosmological solution found in [11] the mass parameter (3) can be computed obtaining (we refer to [14] for details of the computation):

$$m^2 = \frac{\mu(n)\rho^{\cos} + \nu(n)\rho^s}{\rho^{\cos} + \rho^s} R_0, \quad R_0(t) = \frac{4(1+4n)(1+n)}{3t^2},$$

where  $\mu(n)$  and  $\nu(n)$  are rational functions of the exponent  $n$ . In [14] it is shown that the condition  $|mr| \ll 1$  imposes the extremely mild constraint  $n \gg (1/6)R_S^2 R_0 \sim 10^{-25}$ . Moreover, from the properties of the cosmological solution [11] we have  $f_{R0}^2 \rho^{\cos}(t)/\kappa = -2n/(4n+1)$ , from which it follows that condition (5) is incompatible with the previous constraint  $n \gg 10^{-25}$ :

$$\left| \frac{\kappa}{f_{R0}^2 \rho^{\cos}(t)} - 1 \right| \rho^{\cos} = \left( 3 + \frac{1}{2n} \right) \rho^{\cos}(t) \gg \rho^s(r) \rightarrow n \ll \frac{\rho^{\cos}}{2\rho^s} \sim 10^{-33}.$$

We now check the assumption  $|R_1| \ll R_0$ . The previous result shows that we can not rely on the validity of Newtonian approximation. Hence we cannot use the effective gravitational constant  $G$  defined in (6) for the estimate of the ratio  $R_1/R_0$ , so that we resort to Newton's gravitational constant  $G_N = c^4/16\pi\kappa$ . The value of this ratio outside the Sun can be computed from the exterior solution (4) for  $R_1$ , while the result for the interior solution requires a more involved computation, based on a polynomial model of the mass density  $\rho^s$ , that can be found in [14]:

$$\frac{R_1}{R_0} \approx \frac{1+4n}{n(1+n)} \frac{G_N M_S}{r} \quad \text{for } r \geq R_S, \quad \frac{R_1}{R_0} \approx \frac{1}{1+n} \quad \text{for } r < R_S.$$

Though  $|R_1| \ll R_0$  for  $n \gg 1$ , the interior solution shows that non-linear terms in the Taylor expansion of  $f^2(R)$  cannot be neglected, contradicting our assumption at the end of Section 2:

$$f^2(R) = f_0^2 \left[ 1 - n \frac{R_1}{R_0} + \frac{n(n+1)}{2} \left( \frac{R_1}{R_0} \right)^2 - \frac{1}{6} n(n+1)(n+2) \left( \frac{R_1}{R_0} \right)^3 \right] + O \left( \left( \frac{R_1}{R_0} \right)^4 \right).$$

The lack of validity of the perturbative regime leads us to conclude that the model (7) cannot be constrained by this method, so that it remains, in this respect, a viable theory of gravity.

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